Announcements

1) HW 1 due tomorrow, followed by the appearance of HW 2, due next week.

2) On HW1, for characteristic function question, $S \neq \emptyset$.

3) Talk today, CB 2090 Niles Johnson (homotopy theory), 3-4

Theorem: (density of $\mathbb{Q}$) $\forall\ \varepsilon > 0$ and $x \in \mathbb{R}$, $\exists\ y \in \mathbb{Q}$ with

$$|y - x| < \varepsilon.$$

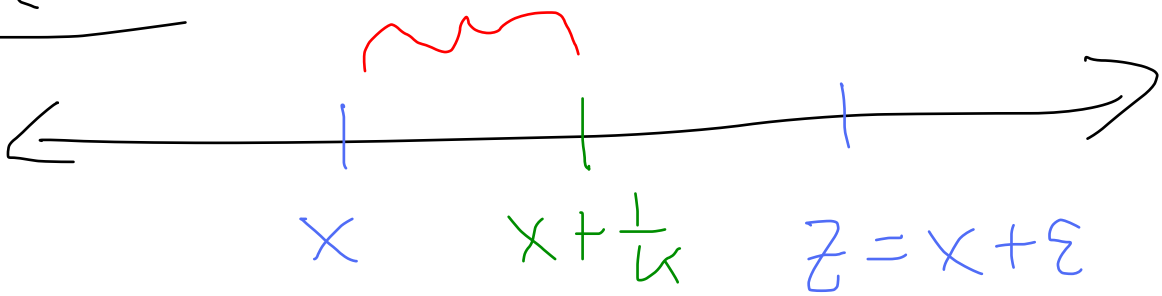
proof: Let $x \in \mathbb{R}$, $\varepsilon > 0$

Let $z = x + \varepsilon$. By the archimedean property, $\exists$ a $k \in \mathbb{N}$, $k > \frac{1}{z-x} = \frac{1}{\varepsilon}$.

Then by cross-multiplying,

$$\varepsilon = z - x > \frac{1}{k}$$

Picture



$\frac{1}{k} < \varepsilon$, so

$$x < x + \frac{1}{k} < x + \varepsilon .$$

Only need to consider x irrational, since $x \in \mathbb{Q} \Rightarrow$ can choose $y = x$.

If $x + \frac{1}{k} \in \mathbb{Q}$, then we set $y = x + \frac{1}{k}$. $|y - x| = y - x = \frac{1}{k} < \varepsilon$, If $x+\frac{1}{k}$ is irrational,

consider the interval

$$\left[kx, k\left(x+\frac{1}{k}\right)\right] = [kx, kx+1]$$

Since $(kx+1)-kx=1$, $\exists$

$n \in \mathbb{N}$, $n \in [kx, kx+1]$.

Then $kx \leq n \leq kx+1$, so

dividing by k,

$$x \leq \frac{n}{k} \leq \frac{kx+1}{k} = x+\frac{1}{k} = z$$

Choose $y = \frac{n}{k} \in \mathbb{Q}$

,

$$|y-x| = y-x = \frac{n}{k} - x$$

$$< \left(x + \frac{1}{k}\right) - x$$

$$= \frac{1}{k} < \varepsilon. \ \square$$

Corollary: $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$

(**Definition**. $T \subseteq S \subseteq \mathbb{R}$. T is dense in S if $\forall \varepsilon > 0$ and $\forall x \in S$, $\exists t \in T$, $|s - t| < \varepsilon$.)

The previous theorem shows $\mathbb{Q}$ is dense in $\mathbb{R}$

proof: Let $\varepsilon > 0$, $x \in \mathbb{R}$.

If x is irrational, there is nothing to prove

Let $x \in \mathbb{Q}$. Appeal to the result, proved in class, that $\sqrt{p}$ is irrational $\forall$ prime numbers p. Consider again $z = x + \varepsilon$ and note $z - x = \varepsilon$.

Apply the Archimedean Property to

$$\left(\frac{1}{z-x}\right)^2$$

to find a $k \in \mathbb{N}$, $k > \left(\frac{1}{z-x}\right)^2$.

Since there are infinitely many primes, $\exists$ a prime p,

$$p \geq K > \frac{1}{(z-x)^2} = \frac{1}{\varepsilon^2}.$$

By cross-multiplying and taking square roots,

$\varepsilon > \frac{1}{\sqrt{p}}$. Since $\sqrt{p}$ is irrational, $\frac{1}{\sqrt{p}}$ is irrational

Claim $y = x + \frac{1}{\sqrt{p}} \notin \mathbb{Q}$.

Recall $x \in \mathbb{Q}$, $\frac{1}{\sqrt{p}} \notin \mathbb{Q}$.

Suppose, by contradiction, that

$x + \frac{1}{\sqrt{p}} \in \mathbb{Q}$, $x + \frac{1}{\sqrt{p}} = \frac{a}{b} \in \mathbb{Q}$

Then subtracting x, $\frac{1}{\sqrt{p}} = \frac{a}{b} - x \in \mathbb{Q}$,

contradiction. End of Claim.

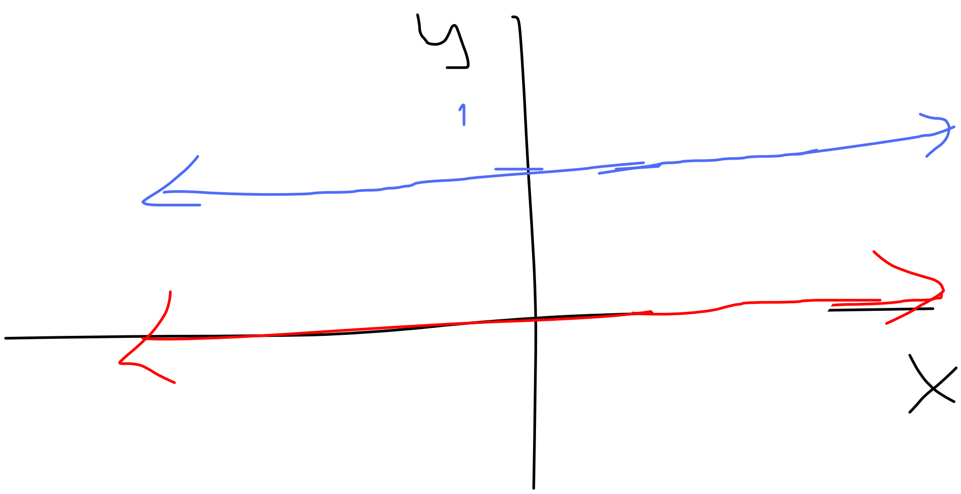
Then $|y - x| = y - x = \frac{1}{\sqrt{p}} < \varepsilon$

$\square$

Example 1: (back to Dirichlet)

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Recall that when we "draw" the graph, it looks like



The previous two density results explain why this picture is the way it is –

both the rationals and the irrationals are so "close" to the real numbers that you can't perceive a difference

Cardinality of Sets

Definition: (injectivity/surjectivity)

Let S, T be sets. A function $f\ S \to T$ is called injective (or one-to-one) if $\forall\ x, y \in S$, whenever $f(x) = f(y)$, then $x = y$.

Such a function is said to be surjective (onto) if $\forall\ y \in T$, $\exists\ x \in S$, $f(x) = y$.

A function that is both injective and surjective is called bijective.

Example 2: $S = T = \mathbb{R}$.

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$ is a bijection

$f: \mathbb{R} \to \mathbb{R}$ $f(x) = \arctan(x)$ is injective but not surjective

(range is $(-\frac{\pi}{2}, \frac{\pi}{2})$)

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \tan(x)$ (x not an integer multiple of $\frac{\pi}{2}$)

$f(n\frac{\pi}{2}) = 0 \quad \forall n \in \mathbb{Z}$.

is surjective but not injective

Definition: (cardinality, countability)

Two sets S and T have the same cardinality if $\exists$ a bijection $f: S \to T$.

S is said to be countable if S has the cardinality of the natural numbers